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FRACTIONAL OPTIMAL CONTROL OF THE TIME FRACTIONAL DIFFUSION SYSTEM

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Abstract

In this article, we study a problem of fractional distributed optimal control for the time fractional diffusion system evolving in a spatial domain $\Omega \subset \mathbb{R}^n$ using distributed bounded controls. We minimize a functional constituted of the deviation between the desired derivative and fractional spatial derivative of order $\alpha \in (0,1)$ and the energy term. We prove the existence of an optimal control solution of the minimization problem. Then, we characterize the control as the solution to an optimality system.

Key words: *Optimal Control; Fractional derivative; Diffusion system; Optimality system.*

I. Introduction

Let Ω be bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ of class C^2. For a given time $T > 0$, we define $G = \Omega \times (0,T)$ and $\partial G = \partial\Omega \times (0,T)$. Consider the fractional diffusion system:

$$\begin{aligned} D_+^{\gamma} z - \Delta z &= u && \text{in } G \\ z &= 0 && \text{on } \partial G \\ I_+^{1-\gamma} z(0^+) &= z^0 && \text{in } \Omega \end{aligned} \tag{1}$$

where $0 < \gamma < 1$, $y^0 \in H^2(\Omega) \cap H_1^0(\Omega)$, the confrol $u \in L^2(G)$. The fractional integral $I_+^{1-\gamma}$ and derivative D_+^{γ} are understood in the Riemann-Lioville sense.

The investigation of fractional diffusion equations and their properties is motivated by their efficient description of anomalous diffusion on fractals. These equations find application in various physical contexts, such as amorphous semiconductors, strongly porous materials, and fractional random walks [1, 2]. Oldham and Spanier [3] established a connection between regular diffusion equations and fractional diffusion equations, introducing a formulation with a first-order spatial derivative and a half-order time derivative. Mainardi and colleagues [4, 5, 6] extended this work by replacing the first time derivative with a fractional derivative of order γ. Agrawal [7] delved into solutions for a fractional diffusion wave equation within a bounded domain, defining the fractional time derivative in the Caputo sense. Through Laplace and finite sine transform techniques, Agrawal obtained a general solution expressed in terms of Mittag-Leffler functions.

In the field of calculus of variations and optimal control for fractional differential equations, limited progress has been made compared to equations with integer time derivatives. Agrawal [8] presented a general formulation and solution scheme for the fractional optimal control problem, where the performance index or the system dynamics, or both, contain at least one fractional derivative term. The fractional derivative was defined in the Riemann–Liouville sense, and the formulation was obtained through the fractional variation principle [9] and the Lagrange multiplier technique. Following a similar approach, Frederico Gastao and Torres Delfim [10] established a Noether-like theorem for the fractional optimal control problem in the sense of Caputo. Agrawal [11] introduced an eigenfunction expansion approach for a class of distributed systems with dynamics defined in the Caputo sense.

We consider the following optimal control problem:

Find the control function $u = u(x,t) \in L^2(G)$ that minimizes the cost functional

$$J(u) = \frac{1}{2}\|D_x^\alpha z - z_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\beta}{2}\|u\|^2_{L^2(0,T;L^2(\Omega))}$$

where D_x^α denotes the fractional spatial derivative of order $\alpha \in (0,1)$, z is a solution of system (1), $z_d \in L^2(\Omega)$ is a desired derivative and $\beta > 0$.

To address this problem, we initially establish the existence and uniqueness of a solution for problem (1) within the space $L^2(G)$. Subsequently, we demonstrate the unique solvability of the optimal control. Finally, by interpreting the first-order optimality condition of the Euler–Lagrange equation through an adjoint problem formulated with a right fractional Caputo derivative, we derive an optimality system for the optimal control. To the best of our knowledge, the contribution presented in this work is new and different in the field of fractional calculus, providing a comprehensive theoretical exploration of the contemplated optimal control and a methodology for its computation.

The subsequent sections of the paper are structured as follows: Section 2 delves into pertinent definitions and preliminary results. In Section 3, we establish the existence and uniqueness of the solution for equation (1). Section 4 elucidates the validity of our optimal control problem and furnishes the optimality system governing the optimal control. The paper concludes with Section 5, where we offer final remarks on our findings.

II. Preliminaries

In this section, we introduce fundamental terminology and articulate key preliminary results those will be employed in the following sections.

Definition 1([12]). *Let $f: \mathbb{R}^+ \to \mathbb{R}$ be continuous function on $\mathbb{R}^+$ and $\gamma > 0$. Then the expression*

$$I_+^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} f(s)\,ds\,, t > 0$$

is called the Riemann-Liounille integral of order γ.

Definition 2([12]) *Let $f: \mathbb{R}^+ \to \mathbb{R}$ be a continuous function on $\mathbb{R}^+$. The Riemann-Liouville fractional derivative of order γ of f is defined by*

$$D_+^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1} f(s)ds \,, t > 0$$

where $\gamma(n-1,n),\ n \in \mathbb{N}$.

Definition 3([12]) *Let* $f: \mathbb{R}^+ \to \mathbb{R}$ *be a continuous function on* $\mathbb{R}^+$. *The left Caputo fractional derivative of order* γ *of* f *is defined by*

$$D_0^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\gamma-1} f^n(s)ds \,, t > 0$$

where $\gamma(n-1,n), n \in \mathbb{N}$.

The Caputo fractional derivative serves as a form of regularization at the temporal origin when compared to the Riemann–Liouville fractional derivative.

Lemma 1 ([12,13]). *Let* $T > 0$, $u \in C^k([0,T])$, $q \in (k-1,k)$, $k \in \mathbb{N}$ *and* $g \in C^1([0,T])$. *Then for* $t \in [0,T]$, *the following properties hold:*

a. $D_+^q g(t) = \frac{d}{dt} I_+^{1-q} g(t), k = 1$.
b. $D_+^q I_+^q g(t) = g(t)$.
c. $I_+^q D_0^q g(t) = g(t) - \sum_{i=0}^{k-1} \frac{t^i}{i!} g^{(i)}(0)$.
d. $\lim_{t\to 0^+} D_0^q\, g(t) = \lim_{t\to 0^+} I_+^q g(t) = 0$.

Henceforth, we consider the right fractional Caputo derivative:

$$\mathcal{D}^\gamma f(t) = \frac{1}{\Gamma(t-\gamma)} \int_t^T (s-t)^{-\gamma} f'(s)ds$$

Lemma 2([14]). *For any* $\phi \in C^\infty(\overline{G})$, *we have*

$$\begin{aligned}\int_0^T \int_\Omega \left(D_+^\gamma z(x,t) - \Delta z(x,t)\right) \phi(x,t)dxdt = \ & \int_\Omega \phi(x,T) I_+^{1-\gamma} z(x,T)dx \\ & - \int_\Omega \phi(x,0) I_+^{1-\gamma} z(x,0^+)dx \\ & + \int_0^T \int_{\partial\Omega} z \frac{\partial \phi}{\partial \nu} d\sigma dt - \int_0^T \int_{\partial\Omega} \frac{\partial z}{\partial \nu} \phi d\sigma dt \\ & + \int_\Omega \int_0^T z(x,t)\left(-\mathcal{D}^\gamma \phi(x,t) - \Delta\phi(x,t)\right)dxdt\end{aligned}$$

Lemma 3([14]). *Let z be the solution of (1). Then for any* $\phi \in C^\infty(\overline{G})$ *such that* $\phi(x,T) = 0$ *in* Ω *and* $\phi = 0$ *on* ∂G, *we have*

$$\begin{aligned}\int_0^T \int_\Omega \left(D_+^\gamma z(x,t) - \Delta z(x,t)\right) \phi(x,t)dxdt = \ & - \int_\Omega \phi(x,0) I_+^{1-\gamma} z(x,0^+)dx \\ & - \int_0^T \int_{\partial\Omega} \frac{\partial z}{\partial \nu} \phi d\sigma dt + \int_0^T \int_{\partial\Omega} z \frac{\partial \phi}{\partial \nu} d\sigma dt \\ & + \int_0^T \int_\Omega z(x,t)\left(-\mathcal{D}^\gamma \phi(x,t) - \Delta\phi(x,t)\right)dxdt\end{aligned}$$

Lemma 4([13]). *Let* $0 < \gamma < 1$. *Let* $g \in L^p(0,T)$, $1 \le p \le \infty$ *and* $\phi: (0,T) \to \mathbb{R}$, *be the function defined by*

$$\phi(t) = \frac{t^{-\gamma}}{\Gamma(1-\gamma)}.$$

Then for almost every $t \in [0,T]$, *the function* $s \mapsto \phi(t-s)g(s)$ *is integrable on* $[0,T]$. *Set*

$$\phi * g(t) = \int_0^t \phi(t-s)g(s)ds.$$

Then $\phi * g \in L^p(0,T)$ *and*

$$\|\phi * g\|_{L^p(0,T)} \leq \|\phi\|_{L^1(0,T)} \|g\|_{L^p(0,T)}.$$

The right function [6, 12, 15] is defined as follows:

$$\Phi_\gamma(x) = \sum_{j=0}^{\infty} \frac{(-x)^j}{j!\,\Gamma(-\gamma j + 1 - \gamma)} = \frac{1}{2i\pi} \int_C s^{\gamma-1} e^{(s - xs^\gamma)}\, ds, 0 < \gamma < 1 \tag{2}$$

where C is a contour which starts and ends at $-\infty$ and encircles the origin once clockwise. The relation between the Wright function and the Mittag-Leffler function is

$$E_\gamma(x) = \int_0^\infty \Phi_\gamma(s) e^{xs}\, ds, 0 < \gamma < 1.$$

That is, $E_\gamma(-x)$ is the Laplace transform of Φ_γ in the whole complex plane.

III. Existence and uniqueness of the solution of (1)

Consider the abstract fractional differential equation in a Banach space $\mathbb{B}$:

$$\begin{aligned} D_+^\gamma z(t) &= Az(t) + h(t), \qquad [0,T] \\ I_+^{1-\gamma} z(0^+) &= z^0, \end{aligned} \tag{3}$$

where $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a Banach space, $0 < \gamma < 1$, $A: D(A) \subset \mathbb{B} \to \mathbb{B}$ is a closed linear operator defined on a dense subset $D(A)$ of the Banach space $\mathbb{B}$, $z_0 \in D(A)$ and $h(t) \in L^2((0,T\ );\ \mathbb{B})$.

In this paper, we consider a Laplace transform of vector-valued functions. The existence and uniqueness of the solution to (2) is considered assuming that A is the generator of a uniformly bounded C_0-semigroup $\big(Q(t)\big)_{t\geq 0}$. That is there exists $K > 0$ such that

$$\sup_{t\geq 0} \|Q(t)\|_{B(\mathbb{B})} \leq K \tag{4}$$

where $(B(\mathbb{B}), \|\cdot\|_{B(B)})$ is the Banach space of all linear bounded operators on $\mathbb{B}$.

Note that, if $(A, D(A)) = (\Delta, H^2(\Omega) \cap H_0^1(\Omega))$, where Δ is the Laplacian operator then A is the generator of a contraction semigroup [30]. Hence (4) is satisfied.

The proof of the existence and uniqueness of (3) is established in [14]. For the sake of completeness, we present the results from [14].

Theorem 1[14]. Let $\frac{1}{2} < \gamma < 1$. Assume that $h \in L^2((0,T);\ \mathbb{B})$ and A is the generator of a C_0-semigroup $\big(Q(t)\big)_{t\geq 0}$ on a Banach space $\mathbb{B}$ satisfying (4). Then for any $z_0 \in D(A)$, problem (3) has a unique solution $z \in L^2((0,T);\ \mathbb{B})$ given by:

$$z(t) = \mathcal{P}_\gamma z_0 + \int_0^t \mathcal{P}_\gamma(t-\tau)h(\tau)\, d\tau$$

where

$$\mathcal{P}_\gamma(t) = \gamma \int_0^\infty s t^{\gamma-1} \Phi_\gamma(s) Q(t^\gamma s) ds \tag{5}$$

with Φ_γ is defined as in (2). Moreover

$$\|z\|_{L^2((0,T);\mathbb{B})} \leq \frac{K}{\Gamma(\gamma)} \left(\sqrt{\frac{2T^{2\gamma-1}}{(2\gamma-1)}} \|z^0\|_{\mathbb{B}} + \sqrt{\frac{2T^{3\gamma}}{\gamma^3}} \|h\|_{L^2((0,T);\ \mathbb{B})} \right). \tag{6}$$

Corollary 1[14]. Let $0 < \gamma < 1$ and $z^0 = 0$. Assume that A is the generator of a C_0-semigroup $\left(Q(t)\right)_{t\geq 0}$ on a Banach space $\mathbb{B}$ satisfying (4). Then problem (3) has a unique solution $z \in L^2((0,T);\ \mathbb{B})$ given by:

$$z(t) = \int_0^t \mathcal{P}_\gamma(t-\tau)h(\tau)\,d\tau \tag{7}$$

Moreover

$$\|z\|_{L^2((0,T);\mathbb{B})} \leq \frac{K}{\Gamma(\gamma)}\left(\sqrt{\frac{T^{3\gamma}}{\gamma^3}}\|h\|_{L^2((0,T);\ \mathbb{B})}\right). \tag{8}$$

Theorem 2[14]. Let $\frac{1}{2} < \gamma < 1$ and $z^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u \in L^2(G)$. Then problem (1) has a unique solution in $L^2(G)$.
Moreover

$$\|z\|_{L^2((0,T);\mathbb{B})} \leq \frac{1}{\Gamma(\gamma)}\left(\sqrt{\frac{2T^{2\gamma-1}}{(2\gamma-1)}}\|z^0\|_{L^2(\Omega)} + \sqrt{\frac{2T^{3\gamma}}{\gamma^3}}\|u\|_{L^2(G)}\right). \tag{9}$$

Corollary 2[14]. Let $0 < \gamma < 1$ and $z^0 = 0$ and $u \in L^2(G)$. Then problem (1) has a unique solution $L^2(G)$. Moreover

$$\|z\|_{L^2(G)} \leq \frac{1}{\Gamma(\gamma)}\left(\sqrt{\frac{T^{3\gamma}}{\gamma^3}}\|u\|_{L^2(G)}\right).$$

Now, consider the backward fractional differential equation:

$$\begin{aligned} -\mathfrak{D}^\gamma p(t) - Ap(t) &= r(t), \qquad t \in [0,T] \\ p(T) &= 0 \end{aligned} \tag{10}$$

where $0 < \gamma < 1$, $r \in L^2((0,T);\ \mathbb{B})$ and A is the generator of a C_0-semigroup $\left(Q(t)\right)_{t\geq 0}$ on a Banach space $\mathbb{B}$.

Theorem 3[14]. Let $0 < \gamma < 1$. Assume that A is the generator of a C_0-semigroup $\left(Q(t)\right)_{t\geq 0}$ on a Banach space $\mathbb{B}$ satisfying (4). Then problem (10) has a unique solution $p \in L^2(G)$ given by:

$$p(t) = \int_0^t \mathcal{P}_\gamma(t-\tau)r(\tau)\,d\tau \tag{11}$$

where $\mathcal{P}(t)$ is the operator defined by (5). Moreover

$$\|p\|_{L^2((0,T);\mathbb{B})} \leq \frac{K}{\Gamma(\gamma)}\left(\sqrt{\frac{T^{3\gamma}}{\gamma^3}}\|r\|_{L^2((0,T);\ \mathbb{B})}\right). \tag{12}$$

IV. Optimal control

In this section we want to find the control function $u = u(x,t) \in L^2(G)$ that minimizes the cost functional

$$J(u) = \frac{1}{2}\|D_x^\alpha z - z_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\beta}{2}\|u\|^2_{L^2(0,T;L^2(\Omega))}$$

where D_x^α denotes the fractional spatial derivative of order $\alpha \in (0,1)$, z is a solution of system (1), $z_d \in L^2(\Omega)$ is a desired derivative and $\beta > 0$.

$$J(u) = \lim_{v \in L^2(G)} J(v) \tag{13}$$

Proposition 1. Assuming the state of the system is described by (1), there exists a unique optimal control, u, that minimizes (13).

Proof: Consider the the sequence $\{u_n\} \subset L^2(G)$, that minimizes (13).

Then,

$$J(u_n) \to \inf_{u \in L^2(G)} J(u). \tag{14}$$

Then $z_n = z(u_n)$ is a solution of (1). That is, z_n satisfies

$$D_+^\gamma z_n - \Delta z_n = u_n, \qquad \text{in } G \tag{15}$$

$$z_n = 0, \qquad \text{on } \partial G \tag{16}$$

$$I_+^{1-\gamma} z_n(x,0) = z^0, \qquad \text{in } \Omega \tag{17}$$

Moreover, from (14), it follows that there exists $C > 0$ independent of n such that

$$\|u_n\|_{L^2(G)} \le C,$$
$$\|z_n\|_{L^2(G)} \le C$$

and follows from (15) that

$$\left\| D_+^\gamma z_n - \Delta z_n \right\|_{L^2(G)} \le C. \tag{18}$$

Thus there exists $u, y, \delta \in L^2(G)$ and subsequences and be extracted from $\{u_n\}$ and $\{z_n\}$ (for simplicity these subsequences still be denoted by $\{u_n\}$ and $\{z_n\}$) such that

$$u_n \rightharpoonup u \ \text{ weakly in } L^2(G), \tag{19}$$

$$z_n \rightharpoonup z \ \text{ weakly in } L^2(G), \tag{20}$$

$$D_+^\gamma z_n - \Delta z_n \rightharpoonup \delta \ \text{ weakly in } L^2(G). \tag{21}$$

Define the set $\mathbb{S}(G) = \{\phi \in C^\infty(G) | \phi|_{\partial\Omega} = 0, \phi(x,0) = \phi(x,T) = 0 \text{ in } \Omega\}$ and $\mathbb{S}'(G)$ denotes the dual of $\mathbb{S}(G)$.

From Lemma 3, it follows that

$$\int_0^T \int_\Omega \left(D_+^\gamma z_n(x,t) - \Delta z_n(x,t) \right) \phi(x,t) dxdt = \int_0^T \int_\Omega z_n(x,t) \big(-\mathfrak{D}^\gamma \phi(x,t) - \Delta \phi(x,t) \big) dxdt$$

for all $\phi \in \mathbb{S}(G)$.

Therefore, from (20) it follows that

$$\lim_{n\to\infty} \int_0^T \int_\Omega \left(D_+^\gamma z_n(x,t) - \Delta z_n(x,t) \right) \phi(x,t) dxdt = \int_0^T \int_\Omega z(x,t) \big(-\mathfrak{D}^\gamma \phi(x,t) - \Delta \phi(x,t) \big) dxdt$$

$$= \int_0^T \int_\Omega \left(D_+^\gamma z_n(x,t) - \Delta z_n(x,t) \right) \phi(x,t) dxdt$$

That is,

$$D_+^\gamma z_n - \Delta z_n \rightharpoonup D_+^\gamma z - \Delta z \text{ weakly in } \mathbb{S}'(G),$$

and hence

$$D_+^\gamma z - \Delta z = \delta \in L^2(G). \tag{22}$$

Thus, from (15), (19), (21) and (22), we can deduce that

$$D_+^\gamma z - \Delta z = u \text{ in } G, \tag{23}$$

$$D_+^\gamma z_n - \Delta z_n \rightharpoonup D_+^\gamma z - \Delta z \text{ weakly in } L^2(G). \tag{24}$$

If $z \in L^2(G)$, then from Lemma 4 it follows that $I_+^{1-\gamma} z \in L^2(G)$. Therefore, we have

$D_+^\gamma z = I_+^{1-\gamma} z \in H^{-1}\big((0,T);\ L^2(\Omega)\big)$ and hence $\Delta z \in H^{-1}((0,T); L^2(\Omega))$ because (22) holds. Thus $z(t) \in L^2(\Omega)$ and $\Delta z(t) \in L^2(\Omega)$. Thus, we can conclude that $z|_{\partial\Omega}$ exists and belongs to $H^{-\frac{1}{2}}(\partial\Omega)$ (see [16]).

Also, we have$\Delta z \in L^2((0,T); H^{-2}(\Omega))$ and hence $D_+^\gamma z = I_+^{1-\gamma} z \in L^2\big((0,T); H^{-2}(\Omega)\big)$ since (22) holds. Thus $I_+^{1-\gamma} z \in L^2(G)$ and $\frac{d}{dt} I_+^{1-\gamma} z \in L^2((0,T); H^{-2}(\Omega))$. Consequently $I_+^{1-\gamma} z$ belongs to $C([0,T], H^{-1}(\Omega))$ (see [17]). That is, $I_+^{1-\gamma} z(x,0)$ exists and exist and belongs to $H^{-1}(\Omega)$.

Now, multiplying (15) by $\phi \in C^\infty(\overline{G})$ with $\phi|_{\partial\Omega} = 0$ and $\phi(x,T) = 0$ on Ω, and integrating by parts over G , we obtain by using Lemma 3,

$$\begin{aligned}&\int_0^T \int_\Omega \Big(D_+^\gamma z_n(x,t) - \Delta z_n(x,t)\Big)\phi(x,t)dxdt \\ &\qquad = -\int_\Omega \phi(x,0) z^0\, dx + \int_0^T \int_\Omega z_n(x,t)\big(-\mathfrak{D}^\gamma \phi(x,t) - \Delta\phi(x,t)\big)dxdt\end{aligned}$$

Using (20) and (24), we have

$$\begin{aligned}&\int_0^T \int_\Omega \Big(D_+^\gamma z(x,t) - \Delta \mathrm{z}(x,t)\Big)\phi(x,t)dxdt + \int_\Omega \phi(x,0) z^0\, dx \\ &= \int_0^T \int_\Omega z(x,t)\big(-\mathfrak{D}^\gamma \phi(x,t) - \Delta\phi(x,t)\big)dxdt.\end{aligned} \tag{25}$$

Integration by parts and Lemma 2 gives us

$$\begin{aligned}&\int_0^T \int_\Omega \Big(D_+^\gamma z(x,t) - \Delta \mathrm{z}(x,t)\Big)\phi(x,t)dxdt + \int_\Omega \phi(x,0) z^0\, dx \\ &\qquad = +\big\langle \phi(x,0), I_+^{1-\gamma} z(x,0^+)\big\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_0^T \Big\langle z, \frac{\partial\phi}{\partial\nu}\Big\rangle_{H^{-\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega)} dt \\ &\qquad + \int_0^T \int_\Omega \Big(D_+^\gamma z(x,t) - \Delta z(x,t)\Big)\phi(x,t)dxdt\end{aligned} \tag{26}$$

for all $\phi \in C^\infty(\overline{G})$ with $\phi|_{\partial\Omega} = 0$ and $\phi(x,T) = 0$ in Ω

where $\langle \cdot,\cdot \rangle_{Y,Y'}$ represents the duality bracket between the spaces Y and Y'.

Hence, from (26), it follows that

$$+\int_\Omega \phi(x,0) z^0\, dx = +\big\langle \phi(x,0), I_+^{1-\gamma} z(x,0^+)\big\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \int_0^T \Big\langle z, \frac{\partial\phi}{\partial\nu}\Big\rangle_{H^{-\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega)} dt$$

for all $\phi \in C^\infty(\overline{G})$ with $\phi|_{\partial\Omega} = 0$ and $\phi(x,T) = 0$ in Ω.

Now, if we select ϕ such that $\frac{\partial\phi}{\partial\nu} = 0$ on $\partial\Omega$, then we obtain

$$I_+^{1-\gamma} z(x,0^+) = z^0(x) \text{ in } \Omega \tag{27}$$

and then,

$$z = 0 \text{ on } \partial\Omega. \tag{28}$$

Thus, from (23), (27) and (28) it follows that $z = z(u)$ is a solution of system (1).

For $\alpha \in (0,1)$, D_x^α is a continuous functional from $H_0^1(\Omega) \to L^2(\Omega)$, we have

$$\lim_{n\to\infty} \int_0^T \|D_x^\alpha z_n(t) - z_d\|_{L^2(\Omega)}\, dt = \int_0^T \|D_x^\alpha z(t) - z_d\|_{L^2(\Omega)}\, dt$$

and J is lower semi-continuous with respect to weak convergence, it follows that

$$J(u) \le \lim_{n\to\infty} \inf J(u_n). \tag{29}$$

Hence

$$J(u) = \lim_{v \in L^2(G)} J(v).$$

The uniqueness of u follows from the strict convexity of J.

Proposition 1. If u is a solution (13), then there exists $p \in L^2(G)$ such that the triplet (u, z, p) satisfies the following optimality system:

$$\begin{cases} D_+^\gamma z - \Delta z = u & \text{in } G \\ z = 0 & \text{on } \partial G \\ I_+^{1-\gamma} z(x, 0^+) = z^0 & \text{in } \Omega \end{cases} \tag{30}$$

$$\begin{cases} -\mathfrak{D}^\gamma p - \Delta p = D_x^\alpha z - z_d & \text{in } G \\ p = 0 & \text{on } \partial G \\ p(T) = 0 & \text{in } \Omega \end{cases} \tag{31}$$

$$u = -\frac{2p}{\beta} \quad \text{in } G \tag{32}$$

Proof: (30) follows from (23), (27) and (28). To prove (31) and (32), we apply Euler-Lagrange technique which will characterize the optimal control u.

We have

$$\frac{d}{d\mu} J(u + \mu\phi)|_{\mu=0} = 0, \qquad \text{for all } \phi \in L^2(G). \tag{33}$$

The state y associated with the control $\phi \in L^2(Q)$ is a solution of

$$\begin{cases} D_+^\gamma y - \Delta \mathrm{y} = \phi & \text{in } G \\ y = 0 & \text{on } \partial G \\ I_+^{1-\gamma} y(x, 0^+) = 0 & \text{in } \Omega \end{cases} \tag{34}$$

After differentiating (33), we have

$$\int_0^T \int_\Omega y(D_x^\alpha z(u) - z_d)\, dxdt + \frac{\beta}{2} \int_0^T \int_\Omega u\phi \, dxdt = 0 \;\forall\, \phi \in L^2(G). \tag{35}$$

Consider the following adjoint state equation to interpret (35):

$$\begin{cases} -\mathfrak{D}^\gamma p - \Delta p = D_x^\alpha z - z_d & \text{in } G \\ p = 0 & \text{on } \partial G \\ p(T) = 0 & \text{in } \Omega \end{cases} \tag{36}$$

Since $D_x^\alpha z - z_d \in L^2(G)$, Theorem 3 and $\left(A, D(A)\right) = (\Delta, H^2(\Omega) \cap H_0^1(\Omega))$ together imply that (36) has unique solution in $L^2(G)$. Hence, multiplying (34) p, ang applying Lemma 3, we obtain

$$\begin{aligned} \int_0^T \int_\Omega (D_+^\gamma z - \Delta z) p \, dxdt &= \int_0^T \int_\Omega (-\mathfrak{D}^\gamma p - \Delta p) z \, dxdt \\ &= \int_0^T \int_\Omega (D_x^\alpha z(u) - z_d) z \, dxdt. \end{aligned}$$

Hence, from (34) and (35), we deduce that

$$\int_0^T \int_\Omega \phi p \, dxdt = -\frac{\beta}{2} \int_0^T \int_\Omega \phi u \, dxdt \;\forall\, \phi \in L^2(Q).$$

Thus,

$$u = -\frac{2p}{\beta} \text{ in } G.$$

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